# Topic 2 — DFT, Z-transform and stability

## 2.1 How do you know if a linear system is stable?

Suppose we are given a description of a linear system:

\[
y(t) = \sum_{i=1}^{N} a_i x(t - i) + \sum_{i=1}^{M} b_i y(t - i)
\]

We would like to know if this system is stable, or might become unstable and have outputs diverging to infinity. Of course, if we know the input \(x(t)\) we can simulate the system and find out. But what do we do if we want to guarantee that the output is stable for _all_ reasonable outputs?

At the end of this class you will be able to answer this question.

To start, we revisit the Fourier series using a complex number notation that will allow for a much simpler representation.

Recall that the Fourier series consists of _a pair_ of basis functions for each frequency:

These two basis functions are closely related to each other. Using complex number representation we can make this relationship explicit.

### 2.2 Complex numbers

In the real numbers, the square root of \(-1\) is not defined. The complex numbers are created by adding the root of \(-1\), denoted by \(i\) to the reals.

The normal notation for a random number is \(z = a + bi\), where \(a\) and \(b\) are regular real numbers. \(a\) is called the _Real_ component of \(z\) and \(b\) is the _Imaginary_ component of \(z\). We can represent this complex number as a vector in the 2D plane whose coordinates are \(a\) and \(b\).

Adding two complex numbers corresponds to vector addition:

\[
(a + bi) + (c + di) = (a + c) + (b + d)i
\]

Taking the product of a complex number with a real number is straightforward:

\[
d(a + ib) = da + idb
\]

The conjugate of a complex number \(z = a + ib\) is the number with the imaginary part reversed: \(\overline{z} = a - ib\). It is easy to show that the imaginary part of \(z + \overline{z}\) is zero.

There is also a _polar_ representation of complex numbers:

\[
z = x + yi = r(\cos(\varphi) + i\sin(\varphi))
\]
Figure 2.1. The complex number plane: On the left are three complex numbers and their location on the plane. On the right, a single complex number expressed in both the real/imaginary coordinates \((x, y)\) and the polar coordinates \((r, \varphi)\).

where the magnitude of the number is \(r = |z| = \sqrt{x^2 + y^2}\) and the phase, angle or argument of the number is

\[
\varphi = \arg(z) = \begin{cases} 
\arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\
\arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\
\arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\
\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\
-\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\
\text{undefined} & \text{if } x = 0 \text{ and } y = 0
\end{cases}
\]

The product of two complex numbers is easy to describe using the polar representation. Suppose \(z_1 = r_1(\cos(\varphi_1) + i\sin(\varphi_1)), z_2 = r_2(\cos(\varphi_2) + i\sin(\varphi_2))\) then

\[
z_1z_2 = r_1r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))
\]

In other words, the resulting magnitude is the product of the input magnitudes and the resulting phase is the sum of the input phases. In particular, if the magnitude of \(z_1\) is equal to one, then taking the product corresponds to a pure phase shift, or rotation, of \(z_2\). On the other hand, if the phase of \(z_1\) is zero then the effect of taking the product is to change the magnitude of \(z_2\) without changing it’s phase.

The exponential function can be extended to the complex numbers as follows

\[
\exp(iv) = \cos(v) + i\sin(v)
\]

In particular, \(e^{2\pi i} = 1, e^{i\pi} = -1\) and \(e^{i\pi/2} = i\). Which means that a complex number can be represented in polar coordinates as \(r \exp(iv) = \exp(\ln r + iv)\). Using this representation the product of two complex numbers is very simple.

\[
\exp(\ln r_1 + iv_1)\exp(\ln r_2 + iv_2) = \exp(\ln r_1 + \ln r_2 + i(v_1 + v_2))
\]
2.3 The Fourier Series using complex numbers

We can write \( \cos(\omega t) \) and \( \sin(\omega t) \) using exponentials of imaginary numbers as follows:

\[
\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})
\]
\[
\sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})
\]

Using these expressions we can rewrite the basis functions of the Fourier Series as follows:

\[
d_c(t) = \sqrt{\frac{1}{T}} \exp\left(\frac{2\pi 0}{T} t\right)
\]
\[
\forall k = 1, 2, 3, \ldots
\]
\[
e_k(t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi k}{T} t\right) = \sqrt{\frac{1}{2T}} \left(\exp\left(\frac{2\pi k}{T} t\right) + \exp\left(-\frac{2\pi k}{T} t\right)\right)
\]
\[
o_k(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi k}{T} t\right) = \sqrt{\frac{1}{2T}} \left(\exp\left(\frac{2\pi k}{T} t\right) + \exp\left(-\frac{2\pi k}{T} t\right)\right)
\]

Using indexes going from \(-\infty\) to \(\infty\) we can now use as the orthonormal basis functions:

\[
k = -\infty, \ldots, \infty, \quad e_k = \frac{1}{\sqrt{T}} \exp\left(-i\frac{2\pi kt}{T}\right)
\]

and the continuous time signal \( s(t) \) can be represented as:

\[
s(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi k}{T} t}
\]

Where the coefficients \( c_k \) are complex numbers.

From this representation it is easy to transition to the Discrete Fourier transform, which transforms complex sequences of length \( N \) to complex sequences of length \( N \). We define the constant

\[
W_N = e^{i\frac{2\pi}{N}}
\]

which correspond to \( 1/N \) of a full rotation. Using \( W_N \) we can express each basis function as \( W_N^k \).

Using this factor we define the Fourier representation of the signal \( s(n) \) as

\[
s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k) W_N^k
\]

Where the complex numbers \( S(k) \) are the Frequency coefficients. This is called the inverse Fourier Transform as it transforms from the frequency coefficients \( S(k) \) to the time coefficients \( s(n) \).

The Forward Fourier Transform maps the time sequence \( s(n) \) to the Frequency sequence \( S(k) \) and is defined as

\[
S(k) = \sum_{k=0}^{N-1} s(n) W_N^{-nk}
\]

Note how similar the Forward and inverse transforms are. By convension we don’t use normalized basis vectors and instead divide the sum by \( N \) when mapping the frequency coefficients to the time domain.

Note that each coefficient is a complex number, which is equivalent to two real valued numbers. The time domain sequence \( s(n) \) is usually real valued, which means that the imaginary coefficients are all zero. On the other hand the imaginary coefficients in \( S(k) \) are usually not zero. If \( s(n) \) is real then the constraint on \( S(k) \) is that \( S(k) \) and \( S(N-k) \) are complex conjugates.
2.4 The transfer function

Suppose we have a linear system of the form

\[ y(n) = \sum_{k=1}^{N} a_k x(n-k) + \sum_{k=1}^{M} b_k y(n-k) \]  

(2.1)

Where \( x(n) \) is the input to the system and \( y(n) \) is the output.

Suppose the input is a sinusoid of the form

\[ x(n) = e^{i\omega n} \]

Where \( \omega = \frac{2\pi k}{N} \), then the output is a sinusoid with the same frequency but with potentially different phase and different magnitude

\[ y(n) = H(\omega) e^{i\omega n} \]

\( H(\omega) \) is called the transfer function of the system. The transfer function defines the relationship between the input sinusoid and the output sinusoid for every frequency. Suppose \( X(\omega) \) and \( Y(\omega) \) are the Fourier transforms of \( x(n) \), \( y(n) \) then

\[ Y(\omega) = H(\omega)X(\omega) \]

We will next learn how to compute the transfer function of a linear system.

2.5 The Z-transform

To characterize the transfer function it is useful to consider an extension of the Fourier transform from the points on the unit circle to the whole complex plane.

The Z transform of a sequence \( x(n) \) is define to be the polynomial

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \]

where \( z \) is a complex variable and \( z^{-1} \) corresponds to shifting the sequence one step into the past. Note that if we set \( z = e^{-i\omega} \) then the Z transform becomes the Fourier transform.

The linear system described in Equation (2.1) can be described using the Z transform as follow:

\[ Y(z) = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{N} a_k x(n-k)z^{-n} + \sum_{n=-\infty}^{\infty} \sum_{k=1}^{M} b_k y(n-k)z^{-n} \]

We can extend the sumesover \( k \) to \( k = -\infty, \ldots, \infty \) by defining \( a_k \) and \( b_k \) as zero outside the ranges \( 1, \ldots, N \) and \( 1, \ldots, M \) respectively. We can then rearrange the sums to get

\[ Y(z) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k x(n-k)z^{-n} + \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} b_k y(n-k)z^{-n} \]

Rearranging the indexes we get

\[ Y(z) = \left( \sum_{k=-\infty}^{\infty} a_k z^{-k} \right) \left( \sum_{n=-\infty}^{\infty} x(n)z^{-n} \right) + \left( \sum_{k=-\infty}^{\infty} b_k z^{-k} \right) \left( \sum_{n=-\infty}^{\infty} y(n)z^{-n} \right) = A(z)X(z) + Y(z)B(z) \]
By taking the ratio between the output and the input (in the frequency space) we get a close form expression for the transfer function

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{A(z)}{1 - B(z)} = \frac{\sum_{k=1}^{N} a_k z^{-k}}{1 - \sum_{k=1}^{M} b_k z^{-k}} = \frac{\sum_{k=0}^{L} a_{L-k} z^{k}}{1 - \sum_{k=0}^{L} b_{L-k} z^{k}} \]

Where \( L = \max(N - 1, M - 1) \) and we define \( a_k, b_k \) to be zero where they were not previously defined.

These are polynomials over the complex numbers, and, unlike polynomials over the reals, polynomials of degree \( d \) over the complex numbers can always be written as a product of \( d \) polynomials of degree one (this is the fundamental theorem of Algebra):

\[ \sum_{i=0}^{N} a_i z^{-i} = \prod_{j=1}^{N} (z^{-1} - r_j) \]

Applying this factorization to the formula of the transfer function in the \( z \) domain

\[ H(z) = \prod_{j=1}^{N} \frac{z - r_j}{z - s_j} \]

Clearly, \( H(r_j) = 0 \) which is why the \( r_j \) are called the zeros of the \( Z \) transform. While \( H(s_j) \) grow to infinity and are called the Poles of the \( Z \)-transform.

Plugging \( z = e^{i\omega} \) gives back the Fourier transform.

### 2.6 Region of convergence

The \( Z \)-transform might not be well defined on parts of the complex plane. In particular, it diverges at the poles. The existence of poles causes divergence also at other points.

A system is causal if its output at time \( n \) depends only on the inputs and outputs before \( n \). The region of convergence (ROC) of the \( Z \) transform for a causal system contains all points whose magnitudes are larger than the pole with the maximal magnitude.

As our main concern is with the Fourier transform, we are interested in whether it is part of the ROC. The system is stable if the unit circle is in the ROC, equivalently, if all the poles are inside the unit circle.

### 2.7 Phase and Magnitude of the transfer function

When considering the system transfer function \( H(\omega) \) it is useful to consider separately the magnitude \(|H(\omega)|\) and the phase \( \arg(H(\omega)) \). The magnitude of the phase transform corresponds to the amplification or attenuation of the frequency component \( \omega \).

The phase corresponds to the shift in time of the frequency component. Sometimes we don’t care about changes in the phase, for example, our hearing cannot detect changes in phases of a sound wave. However, sometimes, for example in beam-forming applications, we need to retain the phase information. We say that a filter is Linear phase if the phase of the transfer function depends linearly on the frequency \( \omega \). This corresponds to shifting the signal and all of frequency components by the same amount of time shift.