1 Linear maps between Euclidean spaces

For the most part, we’ve been viewing matrices in the context of a system of linear equations $Ax = b$. There, the matrix $A$ holds the variable coefficients for the LHS of the linear equations. We also interpreted the column space and null space of $A$ in this context: the column space $\mathcal{R}(A)$ is the set of RHS vectors $b$ for which $Ax = b$ has a solution, and the null space $\mathcal{N}(A)$ is the set of solutions to $Ax = 0$. (There are two other vector important spaces associated with $A$: the row space of $A$ and the left null space of $A$. But these are similar: the row space is the column space of $A^\top$, and the left null space is the null space of $A^\top$. See Section 2.4 for more details.)

In this lab, we’ll take a different view of matrices—specifically, the view of $A \in \mathbb{R}^{m \times n}$ as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$—and here, the column space and null space will have yet another interpretation.

Actually, we’ve already seen matrices in this light: recall, the rotation matrix

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$ (1)

defines a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$. What does this map do? It takes a point $x \in \mathbb{R}^2$ and rotates it counter-clockwise by $\theta$ radians about the origin.

In general, a linear map $f$ from a vector space $V$ to another vector space $W$ is a function $f : V \to W$ with the following properties:

1. $f(u + v) = f(u) + f(v)$ for any $u, v \in V$.
2. $f(cu) = cf(u)$ for any $c \in \mathbb{R}$ and $u \in V$.

For example, the rotation function $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f_\theta(x) = A_\theta x$ is a linear map, because for any $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$,

$$f_\theta(x + y) = A_\theta(x + y) = A_\theta x + A_\theta y = f_\theta(x) + f_\theta(y)$$

and

$$f_\theta(cx) = A_\theta(cx) = cA_\theta x = cf_\theta(x).$$

Indeed, any function defined by $f(x) = Ax$ for some matrix $A \in \mathbb{R}^{m \times n}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, simply because of the properties of matrix-vector multiplication.

Warm-up exercise$^1$. Let $A_\theta$ be the matrix defined in Equation 1. What is the column space of $A_\theta$? What is the null space of $A_\theta$? ■
For any invertible matrix $A \in \mathbb{R}^{n \times n}$, we know that its null space $\mathcal{N}(A) = \{0\}$, and this space has dimension 0. Therefore, the column space of $A$ must have dimension $n$, which means the column space $\mathcal{R}(A)$ is all of $\mathbb{R}^n$. What does this mean in terms of the linear map $f(x) = Ax$?

- We know that the solutions to $Ax = 0$ comprise $\mathcal{N}(A)$. Therefore, since $\mathcal{N}(A) = \{0\}$, the only $x$ for which $Ax = 0$ is 0. In other words, $f(x) = 0$ if and only if $x = 0$.

- The fact that $\mathcal{R}(A) = \mathbb{R}^n$ means that for every $b \in \mathbb{R}^n$, there is some $x \in \mathbb{R}^n$ such that $Ax = b$. In other words, for every $b \in \mathbb{R}^n$, there is some $x \in \mathbb{R}^n$ for which $f(x) = b$. In fact, we know exactly which $x$ gives $f(x) = b$: it is precisely given by $x = A^{-1}b$. Let’s check this: $f(A^{-1}b) = A(A^{-1}b) = (AA^{-1})b = I_m b = b$.

Exercise 1. Let

$$B = \begin{bmatrix} \frac{-\sqrt{11}}{\sqrt{20}} & 0 \\ \frac{-\sqrt{10}}{\sqrt{20}} & \frac{1}{3} \\ \frac{-1}{\sqrt{20}} & \frac{\sqrt{2}}{3} \end{bmatrix}, \quad C = BB^\top,$$

and $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x) = Cx$.

(a) Let $v_1, v_2, \ldots, v_{3641}$ be the columns of the matrix $V$ from lab6data.mat. Plot all of the $v_i$ and $f(v_i)$ in a single figure, using different colors and symbols to distinguish $v_i$ and $f(v_i)$.

(b) What are the dimensions of $\mathcal{N}(C)$ and $\mathcal{R}(C)$? On the figure from the previous part, depict $\mathcal{N}(C)$ (as a line, plane, or whatever it is). Set the camera view angle at $\text{view}(0, 0)$ and the axes at $\text{axis}([-100, 100, -100, 100, -100, 100])$.

(c) The matrix $B^\top$ also defines a map $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by $g(x) = B^\top x$. Plot the $g(v_i)$ in yet another figure, and compare it to the plot of the $f(v_i)$. What is the main difference between $f$ and $g$?

Exercise 2. Rotation maps in $\mathbb{R}^3$ can be defined with respect to any axis. Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ be any vector with $u_1^2 + u_2^2 + u_3^2 = 1$. Then, the matrix $A_{\theta} \in \mathbb{R}^{3 \times 3}$ that rotates a point about the axis $u$ by $\theta$ radians is

$$A_{\theta} = P + \cos(\theta)(I_3 - P) + \sin(\theta)Q$$

where

$$P = uu^\top \quad \text{and} \quad Q = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

\[\text{Answer: } \{0\} = (\theta V)^\prime \mathcal{N}\text{ pur } \mathcal{V} = (\theta V)^\prime \mathcal{V}^\prime\]
Write a script that shows the columns of \( V \) (the matrix in \texttt{lab6data.mat}) rotating about the axis \( u = (-1, +1, -1)/\sqrt{3} \) in increments of \( \theta = \pi/200 \) radians. Fix the camera view angle at \texttt{view(0, 0)} and the axes at \texttt{axis([-100, 100, -100, 100, -100, 100])}. Have the points make a few revolutions about the axis. Include this script with your write-up together with a plot of the points after having rotated \( \pi/4 \) radians from their original positions.

Hints: Each \texttt{plot} command will try to adjust the axes automatically; to keep the axes fixed, issue and \texttt{axis} command after each call to \texttt{plot}. You will also probably want to use \texttt{pause} after each \texttt{plot} command.

In summary, a linear map \( f : \mathbb{R}^n \to \mathbb{R}^m \) given by \( f(x) = Ax \) sends every vector \( x \) in the domain \( \mathbb{R}^n \) to some other vector \( y \) in \( \mathcal{R}(A) \subseteq \mathbb{R}^m \), and the \( x \)'s that get mapped to 0 are those in \( \mathcal{N}(A) \). When \( A \) is invertible, the only vector \( x \) that gets mapped to 0 is \( 0 \). (Note: the vector \( 0 \in \mathbb{R}^n \) always gets mapped to \( 0 \in \mathbb{R}^m \) by a linear function.)

## 2 Linear maps of polynomials

Let \( P_n \) be the vector space of polynomials of degree at most \( n \), i.e. \( P_n = \{ p \in \mathbb{R} \to \mathbb{R} : p(t) = a_0 + a_1 t + \ldots + a_n t^n \text{ for some } a_0, a_1, \ldots, a_n \in \mathbb{R} \} \) (here, \([\mathbb{R} \to \mathbb{R}]\) denotes the set of all functions from \( \mathbb{R} \) to \( \mathbb{R} \); a polynomial is one such function). Addition and scalar multiplication in this space is exactly what you’d expect: for \( p, q \in P_n \) given by \( p(t) = a_0 + a_1 t + \ldots + a_n t^n \) and \( q(t) = b_0 + b_1 t + \ldots + b_n t^n \), and \( c \in \mathbb{R} \),

- \( p + q \) is the polynomial \( [p + q](t) = (a_0 + b_0) + (a_1 + b_1) t + \ldots + (a_n + b_n) t^n \).
- \( cp \) is the polynomial given by \( [cp](t) = (ca_0) + (ca_1) t + \ldots + (ca_n) t^n \).

(Above, we used the notation \( [\text{expr}](t) \) to mean the function given by the expression “\text{expr}” evaluated at \( t \).)

### Warm-up exercise\(^2\)

Give a basis for \( P_n \). What is the dimension of \( P_n \)?

Here are some examples of linear maps on \( P_n \):

- Differentiation: \([f(p)](t) = \frac{d}{dt} p(t)\). The null space of \( f \) is \( \{ p \in [\mathbb{R} \to \mathbb{R}] : p(t) = a_0 \text{ for some } a_0 \in \mathbb{R} \} \), \text{i.e.} the space of constant functions \( P_0 \). The range of \( f \) is \( P_{n-1} \).

- Integration from 0 to \( t \): \([f(p)](t) = \int_0^t p(u) du\). The null space of \( f \) is \( \{0\} \), \text{i.e.} the set containing just the zero function. The \textit{co-domain} of \( f \) is \( P_{n+1} \), although the \textit{range} is only a subspace of \( P_{n+1} \).

- Multiplication by a fixed polynomial \( q \in P_m \): \([f(p)](t) = p(t) q(t)\). For example, if \( q(t) = 1 + 2t^2 \) and \( p(t) = 1 + 2t + t^2 \), then \([f(p)](t) = (1 + 2t^2)(1 + 2t + t^2) = 1 + 2t + 3t^2 + 4t^3 + 2t^4 \).

In this example, the null space of \( f \) is \( \{0\} \), the co-domain is \( P_{n+2} \), although the range is only a subspace of \( P_{n+2} \).

### Exercise 3

Let \( f : P_n \to P_{n+1} \) be given by \([f(p)](t) = \int_0^t p(u) du\).

\(^2\text{Answer: } 1 + u + u^2 + \ldots + u^n.\)
(a) Show that $f$ is a linear map.

(Hint: it suffices to show for any $p, q \in P_n$ and $c \in \mathbb{R}$, that $f(p + cq) = f(p) + cf(q).$

(b) Describe the range of $f$.

(c) Give an example of a vector in $P_{n+1}$ not in the range of $f$.

3 Coordinate representations

You may have already noticed that it is convenient to think of a polynomial $a_0 + at + \ldots + an t^n$ as a vector in $\mathbb{R}^{n+1}$. Specifically, we can intuitively make the following association

$$a_0 + at + \ldots + an t^n \in P_n \leftrightarrow (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}.$$  

Why is this okay? It turns out $(a_0, a_1, a_2, \ldots, a_n)$ is the coordinate representation of the polynomial $a_0 + at + \ldots + an t^n$ with respect to a particular ordered basis, and that basis is $B = \{1, t, t^2, \ldots, t^n\}$. (The order of the basis elements is important, hence the name, so technically we should not use the set notation here.) We call $B$ the standard basis for $P_n$. Note that a different basis for $P_n$ would yield a different coordinate representation in $\mathbb{R}^{n+1}$ (but it would still be a vector in $\mathbb{R}^{n+1}$, since any basis for $P_n$ must have size $n + 1$).

**Warm-up exercise**. Let $S = \{1, 1 + t, 1 + t^2\}$, an ordered basis for $P_2$. What is the coordinate representation of $1 + 2t + t^2$ with respect to the basis $S$? In other words, what coefficients $(s_0, s_1, s_2) \in \mathbb{R}^3$ satisfy $(s_0)1 + (s_1)(1 + t) + (s_2)(1 + t^2) = 1 + 2t + t^2? 

Now that we can represent polynomials in $P_n$ as ordinary coordinate vectors in $\mathbb{R}^{n+1}$, we can look at what it means to multiply these coordinate vectors by a matrix. This is one of the main advantages of the coordinate representation: any linear map can be represented by a matrix with respect to some bases.

We’ll proceed by example, starting with differentiation. Recall: differentiation on $P_3$ is the linear map $f : P_3 \rightarrow P_3$ given by $[f(p)](t) = \frac{dt}{d} p(t)$. For example, if $p(t) = 1 + 2t + 3t^2 + 4t^3$, then $[f(p)](t) = 2 + 6t + 12t^2$. The matrix representation of this linear map, with respect to the standard basis $\{1, t, t^2, t^3\}$, is

$$A_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

Let’s try multiplying the coordinate representation of $1 + 2t + 3t^2 + 4t^3$ with respect to the standard basis (i.e. $v = (1, 2, 3, 4)$) by this matrix $A_f$:

$$A_nv = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ 0 \end{bmatrix},$$  

3Answer: (1 2 3 4).
and \((2, 6, 12, 0)\) is indeed the coordinate representation of \(2 + 6t + 12t^2\) in the same standard basis. (Note: earlier we claimed that the range of \(f\) would be \(P_2\), but it seems here \((2, 6, 12, 0)\) is the coordinate representation of a polynomial in \(P_3\). Indeed, technically every polynomial in the range of \(f\) is a polynomial in \(P_3\), but the coordinate representation of each such polynomial has a zero coefficient for the basis vector \(t^3\). So the range is essentially \(P_2\). Nevertheless, sometimes it will be useful to explicitly think of differentiation on \(P_3\) as a map from \(P_3\) to \(P_2\), so that the resulting matrix representation will be a \(3 \times 4\) matrix, rather than a \(4 \times 4\) matrix.)

How did we come up with this matrix \(A_f\)? The recipe is simple:

Let \(f : V \to W\) be a linear map, \(B\) an ordered basis for \(V\), and \(B'\) an ordered basis for \(W\). For each basis vector \(u \in B\) in order, find the coordinate representation of \(f(u)\) in basis \(B'\). Stack these side-by-side in a matrix, in the same order. The result is the matrix representation of \(f\) with respect to the ordered bases \(B\) and \(B'\).

Let’s use this recipe to re-derive the matrix \(A_f\) from above. We’ll use the standard bases for both the domain and co-domain of \(f\), i.e. \(\{1, t, t^2, t^3\}\). (Note that in general, the domain and co-domain could have different ordered bases.)

- \(f(1) = 0\) has coordinate representation \((0, 0, 0, 0)\).
- \(f(t) = 1\) has coordinate representation \((1, 0, 0, 0)\).
- \(f(t^2) = 2t\) has coordinate representation \((0, 2, 0, 0)\).
- \(f(t^3) = 3t^2\) has coordinate representation \((0, 0, 3, 0)\).

Stacking these vectors as columns side-by-side indeed gives \(A_f\).

This is a nice and useful property about linear maps: to determine how a map \(f : V \to W\) behaves when applied to any vector \(v \in V\), we just need to know how \(f\) behaves when applied to each vector in a basis for \(V\).

**Exercise 4.** What are the matrix representations for the following linear maps with respect to the standard bases? What are their null spaces and column spaces?

(a) \(f : P_3 \to P_4\) given by \([f(p)](t) = \int_0^t p(u) du\).
(b) \(f : P_3 \to P_5\) given by \([f(p)](t) = (1 + 2t^2) p(t)\).
(c) \(f : P_4 \to P_4\) given by \([f(p)](t) = \frac{d^2}{dt^2} p(t)\).

**Exercise 5.** Let \(M \in \mathbb{R}^{2 \times 3}\) be the matrix representation of differentiation in \(P_2\) (a map from \(P_2\) to \(P_1\)), and let \(N \in \mathbb{R}^{3 \times 2}\) be the matrix representation of integration in \(P_1\) (a map from \(P_1\) to \(P_2\)). These are both with respect to the standard bases. Compute \(MN\). Give a simple interpretation for the result. Compute \(NM\). Explain why the result is different from \(MN\).
Exercise 6. Let \( B = \{1, t, t^2, t^3, t^4\} \) be the standard ordered basis for \( P_4 \), and

\[
B' = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} t, \sqrt{\frac{5}{8}} (3t^2 - 1) \right\}
\]

be an ordered basis for \( P_2 \) (these are the Legendre polynomials up to order 2). Let \( f : P_4 \rightarrow P_2 \) be the linear map defined by the matrix

\[
A_f = \begin{bmatrix}
\sqrt{2} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{5} \\
0 & \sqrt{\frac{2}{3}} & 0 & \frac{\sqrt{5}}{5} & 0 \\
0 & 0 & \frac{2\sqrt{10}}{15} & 0 & \frac{4\sqrt{10}}{35}
\end{bmatrix}
\]

with respect to \( B \) and \( B' \).

To determine \( f(p) \) for any \( p \in P_4 \), first determine the coordinate representation of \( p \) with respect to \( B \) (the basis for the domain of \( f \)); let \( v \) be this vector. Then, compute \( A_f v \); let \( w \) be the resulting vector. The vector \( w \) is the coordinate representation of \( f(p) \) with respect to \( B' \) (the basis for the co-domain of \( f \)). Now \( f(p) \) is simply the linear combination of the basis vectors in \( B' \) prescribed by the entries in \( w \).

(a) For each \( p \in B \), write \( f(p) \) as a linear combination of the basis vectors in \( B' \) (you may call them \( q_0, q_1, \) and \( q_2 \)). Then, for each \( p \in B \), plot \( p(t) \) and \( [f(p)](t) \) over \( t \in [-1, 1] \), both in the same figure. Be sure to label which is which, unless they overlap exactly.

(b) In the previous part of this exercise, for some \( p \in B \), the plots of \( p(t) \) and \( [f(p)](t) \) overlapped exactly. Explain why this happened, and why it didn’t happen for the other \( p \in B \).

(c) Let \( a \in P_4 \) be the polynomial given by \( a(t) = 1 + 2t + t^3 \), and \( b \in P_4 \) be the polynomial given by \( b(t) = 1 - 2t^2 + t^4 \). Plot \( a(t) \), \( [f(a)](t) \), \( b(t) \), and \( [f(b)](t) \) over \( t \in [-1, 1] \), all in the same figure. Be sure to label which is which.

Suppose we want to switch between two bases \( B \) and \( B' \) for the same vector space \( V \). This is simply the identity map \( f : V \rightarrow V \) given by \( f(v) = v \), but using one ordered basis \( B \) for the domain, and another \( B' \) for the co-domain. Therefore, the same recipe given above can be used to determine a change-of-basis matrix.

Exercise 7. Let \( B = \{1, t, t^2\} \) and \( B' = \{1, 1 + t, 1 + t^2\} \) be ordered bases for \( P_2 \).

(a) Find \( A_1 \), the change-of-basis matrix from \( B \) to \( B' \) (i.e. using \( B \) as the basis for the domain, and \( B' \) as the basis for the co-domain).

(b) Find \( A_2 \), the change-of-basis matrix from \( B' \) to \( B \).

(c) What do you notice about \( A_1 \) and \( A_2 \) (in relation to each other)?

Any invertible matrix can be thought of as a change-of-basis matrix, although the basis it changes to may not be particularly meaningful. But some bases are immensely important; one particular example is the Fourier basis, and the change-of-basis matrix to that basis is the Fourier matrix.