1. (10 points) Let $M$ be a matrix and $v$ be a vector

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix}, \quad v = (v_1, v_2, v_3)$$

Write down the condition for $v$ to be an eigenvector of $M$ with eigenvalue $\lambda$. You can use the sum notation ($\sum_i a_i b_i$) to make your expressions shorter.

2. (20 points) Suppose that $x = (x_1, x_2, x_3)$ is a random variable whose covariance matrix is $M$. What conditions should the matrix $A$ satisfy so that the random vector $Ax^T$ has a diagonal covariance matrix?

3. (25 points) Suppose that the matrix $P$ defines the joint distribution of two discrete random variables, $X$ and $Y$, which achieve the values $\{1, 2, 3\}$

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix}$$

Where $p_{i,j} = P(X = i \text{ and } Y = j)$.

Using either matrix notation or sum notation, write the equations that correspond to each of the following conditions or quantities.

(a) (5 points) $X$ and $Y$ are independent.

(b) (5 points) $X$ and $Y$ are identically distributed.

(c) (5 points) The means $E(X), E(Y)$.

(d) (10 points) The covariance matrix for $X$ and $Y$ (it is a two by two matrix.)

4. (20 points) Suppose that $t_1 < t_2 < t_3 < ...$ is a sequence of real numbers generated by a Poisson process with the property that the probability of $k$ events occurring in the time segment $[2, 5]$ is $p$. Give the expression for the probability that the gap between the first and second events $t_2 - t_1$ is smaller than $1$.

5. (10 points) Let $x = (x_1, x_2, \ldots, x_n)$ be a random binary sequence generated IID where $x_i = 1$ with probability $p = 1/1000$ and $x_i = 0$ with probability $1 - p$. Suppose $n = 1,000,000$. Give the best upper bound you can on the probability that $\sum_{i=1}^{n} x_i \leq 500$.

6. (10 points) Suppose that we are given $k$ binary sequences, $s_1, \ldots, s_k$ each of which is 1,000,000 bits long. Suppose $x$ is a random sequence generated IID with equal probabilities: $x_i = 1$ with probability $1/2$ and $x_i = 0$ with probability $1/2$. We define $k$ additional sequences by taking the exclusive or (XOR) of $x$ and each of the $k$ sequences. Let $n_i$ be the number of 1’s in the sequence $(x \text{ XOR } s_i)$. Give the smallest upper bound you can on the probability that $n_i \leq 499,500$ for at least one value of $i$ between 1 and $k$. 


7. (10 points) Under the same conditions on the generation of \( x \) as question 5, give an estimate for the probability that the number of heads is in the range \([1000, 1010]\). Your answer should not be the sum of binomials which gives the exact probability. Instead, use the central limit theorem to come up with a simple approximation.

8. You can answer any subset of the following questions (total 40 points)

(a) (5 points) Give a short and precise definition for the significance of a statistical test.

(b) (6 points) Suppose that you administer \( k \) tests for the same null hypothesis, each with the same significance value \( \delta \), and all using the same single data set, and suppose that all of the tests reject the null hypothesis. With what significance can you reject the null hypothesis?

(c) (8 points) Same as in (b) with the difference that each test rejects a different null hypothesis. With what significance can you reject all of the null hypotheses?

(d) (9 points) Same as in (b) with the difference that each test is performed on a separate, independently drawn, data set. With what significance can you reject the null hypothesis?

9. (5 points) Write the definition of a probability distribution over the integers.

10. (5 point) Write the definition of a probability density over the reals.

11. (5 points) Write expressions for the mean and variance of a real valued random variable whose distribution is a mixture of a density over the reals and a distribution over the integers.

12. Suppose a decimal sequence \( x_1, x_2, \ldots, x_n \) is generated by the IID distribution

\[
\begin{align*}
P(x = 0) &= 1/55, & P(x = 1) &= 2/55, & P(x = 2) &= 3/55, & P(x = 3) &= 4/55, & P(x = 4) &= 5/55, \\
P(x = 5) &= 6/55, & P(x = 6) &= 7/55, & P(x = 7) &= 8/55, & P(x = 8) &= 9/55, & P(x = 9) &= 10/55 \\
\end{align*}
\]

(a) (5 points) Calculate the entropy for the distribution of a single digit.

(b) (15 points) Compute the binary Huffman code for a single digit. Write down the tree and the binary code for each of the digits. Calculate the expected code length for a sequence of length \( n \).

(c) (10 points) Write the expression for the expected code length for a sequence of length \( n \) if arithmetic coding is used instead of Huffman coding.