A simple Lorenz circuit and its radio frequency implementation

Jonathan N. Blakely, Michael B. Eskridge, and Ned J. Corron

U.S. Army Research, Development, and Engineering Command, AMSRD-AMR-WS-ST,
Redstone Arsenal, Alabama 35898

(Received 15 November 2006; accepted 16 March 2007; published online 21 May 2007)

A remarkably simple electronic circuit design based on the chaotic Lorenz system is described. The circuit consists of just two active nonlinear elements (high-speed analog multipliers) and a few passive linear elements. Experimental implementations of the circuit exhibit the classic butterfly attractor and the hysteretic transition from steady state to chaos observed in the Lorenz equations. The simplicity of the circuit makes it suitable for radio frequency applications. The power spectrum of the observed oscillations displays a peak frequency as high as 930 kHz and significant power beyond 1 MHz. [DOI: 10.1063/1.2723641]

The chaotic Lorenz system appears frequently in theoretical studies of chaos communications. The typical Lorenz wave form is an example of a binary symmetric antipodal communication signal and information can be efficiently encoded into the symbolic dynamics of the Lorenz system. Recently, a matched filter for a Lorenz-like wave form was derived. Despite this theoretical progress, a practical high-frequency electronic implementation of the Lorenz system has yet to be reported. In this paper, we describe an electronic circuit that is simpler than previously reported Lorenz circuits, oscillates at radio frequencies, and reproduces the transition to chaos that occurs in the Lorenz model.

The Lorenz system was originally developed as a simplified mathematical model of atmospheric instabilities and does not derive from a basic circuit topology. However, the Lorenz equations have been implemented electronically by various authors using audio frequency components. Typically, these circuits contain several operational amplifiers, which perform linear operations (e.g., integration and summation), as well as a couple of integrated circuits that perform the nonlinear operations (i.e., multiplication). The relatively large number of active components makes it difficult to directly generalize these designs to high frequencies. A monolithic audio frequency implementation has also been reported. Another approach is to use a digital signal processor and digital-to-analog converters. Other researchers have developed systems that resemble the Lorenz oscillator to a greater or lesser extent but are simpler to implement. However, each of these systems differs qualitatively in some way from the Lorenz equations. Also, many of these designs predate the commercial introduction, in the mid 1990s, of accurate, wideband analog multipliers. Here we describe a new circuit that (1) contains just two active components, (2) oscillates at radio frequencies, and (3) reproduces the hysteretic transition from steady state to chaos observed in the Lorenz equations as a bifurcation parameter is varied.

To obtain a circuit implementation of the Lorenz system, we begin by transforming the equations into a suitable form. The original Lorenz equations are

\[
\begin{align*}
\frac{dx}{dt} &= \sigma (-x + y), \\
\frac{dy}{dt} &= Rx - y - xz, \\
\frac{dz}{dt} &= xy - bz,
\end{align*}
\]

where \(x\), \(y\), and \(z\) are dynamic variables and \(\sigma\), \(R\), and \(b\) are fixed parameters. In order to represent \(x\), \(y\), and \(z\) as voltages, the range of variation of these quantities must be reduced as they would otherwise exceed the power supply voltages of typical integrated circuits. This is achieved by a rescaling defined by \(x = \eta X\), \(y = \eta Y\), and \(z = \epsilon Z\), where \(\eta\) and \(\epsilon\) are fixed scale factors. It is also convenient to rescale time so that the oscillation time scale of the circuit can be directly tuned. Thus, we let \(t = T\tau\), where \(T\) is a unit of time. The resulting form of the Lorenz equations is then

\[
\begin{align*}
T\frac{dX}{d\tau} &= \sigma (-X + Y), \\
T\frac{dY}{d\tau} &= RX - Y - \epsilon XZ, \\
T\frac{dZ}{d\tau} &= \frac{\eta^2}{\epsilon} XY - bZ.
\end{align*}
\]

This linear transformation preserves all qualitative features of the original system.

We now introduce a circuit design based on Eqs. (2). The circuit, shown schematically in Fig. 1(a), consists of two active multipliers (\(M1\) and \(M2\)), capacitors (\(C_1\), \(C_2\), and \(C_3\)), resistors (\(R_1\) and \(R_2\)), and a dc voltage source (\(V_{dc}\)). The circuit’s simplicity is due to two design choices that distinguish it from previous Lorenz circuit designs. First, linear mathematical operations are not performed by the opera-
tional amplifiers commonly used in audio frequency Lorenz circuits. Rather, summation is performed by adding currents at nodes according to Kirchhoff’s current law and integration is performed by charging the capacitors. Since summation is performed on currents, the voltage outputs of the multipliers must be converted into currents; thus, the second design choice is to configure the analog multipliers as current output devices.

Commercially available analog multiplier circuits usually produce an output voltage that is proportional to the product of two differential voltage inputs. However, many also feature a third voltage input that is summed with the product of two differential voltage inputs. However, the model is a good starting point for further observations. However, the model is a good starting point for further observations. However, the model is a good starting point for determining what dynamics may be expected in a physical implementation.

We implemented this design on a multipurpose, through-hole PC board (Radio Shack 276-150). Our experimental circuit uses high-speed analog multipliers (Analog Devices, AD734, small signal bandwidth 10 MHz, \( \mu = 10 \, \text{V} \)). The multipliers are supplied power at ±15 V and are soldered directly to the PC board without a socket. The voltage \( V_{\text{dc}} \) was supplied by a bench-top variable dc supply (Agilent E3630A) with an output range of 0–6 V.

In order to explore the full range of dynamics accessible to this circuit, we experimented with many different sets of passive linear components. Here we focus on two particular sets listed in Table I. We refer to these two implementations as circuit A and circuit B. We first consider circuit A. We observed the oscillation of this circuit using a Tektronix TDS7254B oscilloscope and compared it to numerical simulations. All resistors have a 5% tolerance. All capacitors have a 20% tolerance.

![Image](94x598 to 238x743)

**FIG. 1.** (a) Schematic diagram of a simple Lorenz circuit. Analog multipliers M1 and M2 are configured as current output devices. (b) Schematic diagram of a generic analog multiplier M1 configured as a current output device.

\[
\frac{dv_x}{dt} = -\frac{V_Y}{C_1 R_1} v_x + \frac{V_{\text{dc}}}{C_1 R_2} u_R M_1 Z, \quad (5)
\]

where \( v_Y, v_x, \) and \( v_z \) are the voltages over the capacitors \( C_1, C_2, \) and \( C_3, \) respectively. Equations (5) are equivalent to Eqs. (2) with

\[
v_x = X, \quad v_y = Y, \quad v_z = Z.
\]

\[
\sigma = \frac{C_2}{C_1}, \quad b = \frac{R_1 C_2}{R_2 C_3}, \quad R = \frac{V_{\text{dc}} R_1}{u_R M_1} + 1, \quad (6)
\]

\[
T = R_1 C_2, \quad \epsilon = \frac{R_1}{u_R M_2}, \quad \eta = \frac{R_1}{u_R M_2 R_1 M_2} \sqrt{\frac{C_2}{C_3}}.
\]

Since the circuit model [Eqs. (5)] contains two more parameters than the rescaled Lorenz equations [Eqs. (2)], there is no unique set of component values required to implement a given set of Lorenz parameters. Thus, a circuit designer has considerable freedom to choose components that satisfy other design constraints in a particular application. Since this model neglects parasitic reactances, the finite bandwidth of active components, and other nonideal effects, we do not expect exact quantitative correspondence with experimental observations. However, the model is a good starting point for determining what dynamics may be expected in a physical implementation.

![Image](94x598 to 238x743)

**TABLE I.** Values of passive components in experimental circuit implementations. All resistors have a 5% tolerance. All capacitors have a 20% tolerance.

<table>
<thead>
<tr>
<th>Component name</th>
<th>Circuit A Component value</th>
<th>Circuit B Component value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_1 )</td>
<td>6.2 kΩ</td>
<td>6.2 kΩ</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>5.1 kΩ</td>
<td>5.1 kΩ</td>
</tr>
<tr>
<td>( R_{M_1} )</td>
<td>( 560 , \Omega )</td>
<td>( 560 , \Omega )</td>
</tr>
<tr>
<td>( R_{M_2} )</td>
<td>( 56 , \Omega )</td>
<td>( 56 , \Omega )</td>
</tr>
<tr>
<td>( C_1 )</td>
<td>( 200 , \text{pF} )</td>
<td>( 4 , \text{pF} )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( 2 , \text{nF} )</td>
<td>( 39 , \text{pF} )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( 820 , \text{pF} )</td>
<td>( 15 , \text{pF} )</td>
</tr>
</tbody>
</table>

\[
\frac{dv_y}{dt} = \frac{1}{C_2 R_1} \left( \frac{V_{\text{dc}}}{u C_2 R_{M_2}} v_x - \frac{V_{\text{dc}}}{C_2 R_{M_2}} v_x u_R M_1 \right) v_y - \frac{v_x v_y}{u C_2 R_{M_2}}.
\]
a sample rate of 12.5 megasamples per second with 8-bit vertical resolution. At this rate, the wave form is significantly oversampled. The time series has been low pass filtered with a cutoff frequency of 1.25 MHz to reduce measurement and quantization noise. The measured and simulated attractors are quite similar. Circuit A displays a Lorenz-like butterfly attractor with approximately the same range of variation of the voltages \( v_x \) and \( v_z \) as predicted by the model. A notable difference between the experimental and simulated attractors is in the openness of the lobes. At the heart of each lobe is a fixed point. The trajectory of the simulated attractor approaches these fixed points much more closely than does that of the experimental circuit. In this respect, the experimental circuit may more closely resemble a model with a smaller value of \( V_{dc} \).

Figure 3 shows the \( v_x \) components of these same experimental and simulated time series. The time scale of the oscillations is \( \sim 10 \, \mu \text{s} \), an order of magnitude faster than previous audio frequency Lorenz circuits.\(^{1,10-15} \) Close comparison of the time series in Figs. 3(a) and 3(b) reveals the fact that circuit A oscillates slightly slower than predicted by the model. However, we still consider the data in Figs. 2 and 3 to be reasonable confirmation of the circuit design.

A primary motivation for investigating this circuit design is the suitability for high-speed implementation. According to the model, the speed of the oscillation can be increased by simply scaling the three capacitances by the same factor. We found experimentally that the system ceased to oscillate if the capacitors were too small and, in order to regain oscillation, we had to reduce the resistance \( R_{M2} \). We attribute this to the finite slew rate of the multiplier, since reducing \( R_{M2} \) allows the multiplier output to produce the same current with a smaller voltage change. This design approach led us to the parameters of circuit B shown in Table I, the implementation with the fastest dynamics we were able to observe using the AD734 multiplier and through-hole circuit board construction.

Typical performance of this faster circuit implementation is illustrated by Fig. 4, which shows (a) the \( v_x v_z \) phase-space projection and (b) the power spectrum of the voltage \( v_z \) when \( V_{dc}=5.21 \, \text{V} \). Figure 4 is based on a \( 50 \times 10^3 \) point time series sampled at a rate of 625 megasamples per second with 8-bit vertical resolution. The data was low pass filtered with a cutoff frequency of 10 MHz to reduce measurement and quantization noise. The attractor appearing in the phase-space projection has the desired butterfly shape. The power spectrum contains a peak at 930±2.5 kHz with a tail that stretches out well past 1 MHz.

Although this circuit produces a Lorenz-like wave form, it no longer quantitatively matches with simulations of Eq. (5). For example, the bifurcations do not occur near the values of \( V_{dc} \) predicted by the model. Thus, it is reasonable to question in what respects it does or does not behave like the Lorenz system. We now show that the circuit displays a transition to chaos similar to that of the Lorenz equations. In the Lorenz system with the usual parameters \( \sigma=10 \) and \( b=8/3 \), there are two symmetric stable fixed points when \( 1<R<24.06 \). When \( 24.06 < R < 24.76 \), the chaotic set is also stable, so the final state of the system depends on which attractor basin happens to contain the initial condition. Finally, when \( R \approx 24.76 \), the symmetric fixed points lose stability and the chaotic set is the only remaining attractor. The multistability of the system over the range \( 24.06 < R < 24.76 \) results in a hysteretic transition as \( R \) is varied. This transition is depicted in Fig. 5(a), where we plot the mean value of the \( z \) variable as a function of \( R \). The chaotic

FIG. 2. (a) Simulated and (b) experimental phase-space projections of the attractor of circuit A (see Table I for component values).

FIG. 3. (a) Simulated and (b) experimental time series of \( v_x \) from circuit A (see Table I for component values).
FIG. 4. (a) Butterfly attractor and (b) power spectrum of \( v_z \) component measured from circuit B (see Table I for component values).

FIG. 5. Hysteretic transition to chaos observed in both (a) numerical integration of the original Lorenz equations and (b) measurements of circuit B (see Table I for component values) as a bifurcation parameter is varied.

and steady-state solutions have different mean values so the solution branches can be easily separated. At each \( R \) value, we use the final state of the simulation at the previous \( R \) as an initial condition. This keeps the system in the basin of the fixed point as \( R \) is increased through the transition and in the basin of the chaotic set as \( R \) is decreased. Since \( V_{dc} \) is proportional to \( R \), the same transition should be displayed by the circuit as \( V_{dc} \) is increased from a small value. In fact, a similar hysteretic transition is observed in the experimental circuit as shown in Fig. 5(b). Here we plot the mean value of the voltage \( v_z \) as a function of the voltage \( V_{dc} \). At each point we average over a 4 x 10^6 point time series sampled at 500 megasamples per second.

In this paper, we have described a new circuit design based on the Lorenz equations that is simple enough for high-frequency applications. Our fastest experimental implementation displayed Lorenz-type chaotic oscillations with a peak frequency near 1 MHz. The primary limiting factor on the speed of oscillation is the slew rate \( (450 \text{ V}/\mu\text{s}) \) of the AD734 analog multipliers. We note that faster multipliers are commercially available (such as the AD835 with a 1000 V/\mu s slew rate), but taking advantage of these devices requires a more sophisticated printed circuit layout. Many important communication signals have a bandwidth comparable to or smaller than 1 MHz such as analog video, RS-232, and Bluetooth. In the case of analog video (NTSC), the bandwidth is 4.2 MHz. Chaos may prove useful in scrambling wireless analog video links in situations where digital encryption is too costly in terms of power and weight. Alternatively, it may be the case that faster oscillations are necessary for practical chaotic communications applications. A number of simple fast chaotic circuits have already appeared in the literature (e.g., see Refs. 21–25, and references therein). However, not all chaotic systems are equally well suited for all applications. For example, most very fast chaotic circuits are delay dynamical systems. These devices might work well for chaotic scrambling. However, such systems are poorly suited to symbolic dynamics-based communications. If Lorenz chaos faster than that reported here is needed, our results may at least inform efforts to discover faster circuit designs, particularly with respect to the reduced reliance on active components. For example, we note that it has been suggested that chaotic circuit designs should be based on an active sinusoidal oscillator coupled to a passive nonlinearity. It has been observed that the multiplicative nonlinearities of the Lorenz system do not fit well with this paradigm. We believe the results reported here demonstrate an approach outside that paradigm enabling a simple Lorenz circuit implementation using a small number of active components.

The authors gratefully acknowledge the helpful comments of Dr. Scott T. Hayes and Krishna Myneni.