In this lab, you will use Matlab to visualize solutions to systems of linear equations with two or three variables. The goal is to develop some intuition about the geometry of linear equations.

1 The row view

We begin with a system of two linear equations with two variables \((x_1\text{ and } x_2)\).

\[
\begin{align*}
2x_1 - x_2 &= 1 \\
x_1 + x_2 &= 5
\end{align*}
\]

(1)

The row view of this system specifies two lines in two-dimensional Euclidean space \(\mathbb{R}^2\). The first line consists of all points \((x_1, x_2)\) that satisfy the equation \(2x_1 - x_2 = 1\). There are infinitely many such points, so we’ll just plot a few hundred of them; Matlab will interpolate in-between.

```matlab
axis([-10,10,-30,20]);
x1 = -10:0.05:10;
x2 = 2*x1-1; % Solve for x2
plot(x1,x2,'b'); % blue
hold on;
```

(We can also plot the second line similarly.)

```matlab
plot(x1,5-x1,'r--'); % red dashed
hold off;
```

If you squint hard enough (or use the data cursor), you’ll see that the only point the two lines have in common is \((x_1, x_2) = (2,3)\). See Figure 1.

Things are much more interesting in three-dimensional Euclidean space \(\mathbb{R}^3\), so we’ll now consider a system of three linear equations with three
variables \((x_1, x_2, \text{ and } x_3)\).

\[
\begin{align*}
2x_1 & + x_2 + x_3 = 5 \\
4x_1 & - 6x_2 = -2 \\
-2x_1 & + 7x_2 + 2x_3 = 9
\end{align*}
\]  

(2)

Each equation determines a plane in \(\mathbb{R}^3\); the first plane consists of all points \((x_1, x_2, x_3)\) that satisfy the equation \(2x_1 + x_2 + x_3 = 5\). Again, there are infinitely many such points, so we’ll just plot a few thousand of them to get a rough picture. (It is a bit annoying to plot things in \(\mathbb{R}^3\) on a two-dimensional display, so we’ve provided some Matlab functions to do most of the grunt work. Type in each command one at a time, so you see which plane corresponds to each equation.)

```matlab
axis([-20,20,-20,20,-100,100]);
drawplaneq([2,1,1,5]); % coefficients and RHS
hold on;
drawplaneq([4,-6,0,-2]); % don’t forget the 0
drawplaneq([-2,7,2,9]);
hold off;
```

You can view the planes from a different angle by rotating the figure. Unfortunately, it is still a bit difficult to see which points these planes have in
common. See Figure 2.

Let's just deal with two equations at a time. The first two planes share infinitely many points in common, but these points all lie on a line in $\mathbb{R}^3$. So let's plot this line together with the third plane.

\begin{verbatim}
axis([-20,20,-10,10,-100,100]);
drawlineeq([2,1,1,5],[4,-6,0,-2]);
hold on;
drawplaneeq([-2,7,2,9]);
\end{verbatim}

Try zooming-in and rotating the figure so that you can see where the line intersects the plane (Figure 3). They should intersect at a single point $(x_1, x_2, x_3) = (1, 1, 2)$. To verify this, plot the point of intersection.

\begin{verbatim}
drawpoint([1;1;2], 'r.');
hold off;
\end{verbatim}

2 The column view

The column view of the system of equations in (2) is a single vector equation.

\[
\begin{bmatrix}
2 \\
4 \\
-2
\end{bmatrix} x_1 + \begin{bmatrix}
1 \\
-6 \\
7
\end{bmatrix} x_2 + \begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix} x_3 = \begin{bmatrix}
5 \\
-2 \\
9
\end{bmatrix}
\] (3)
Figure 3: Line specified by first two equations in (2) and plane specified by last equation in (2). A red dot is located at the point of intersection.

The coefficients of the variables and the right-hand side (RHS) are three-dimensional \((column)\) vectors. We can visualize a vector in Euclidean space as an arrow emanating from the origin.

```matlab
axis([0,2,-10,10,-5,10]);
drawvector([2;4;-2], 'r'); % red
hold on;
drawvector([1;-6;7], 'g'); % green
drawvector([1;0;2], 'b'); % blue
drawvector([5;-2;9], 'k'); % black
hold off;
```

Try rotating the figure around if you find it difficult to discern all four vectors. See Figure 4.

The row view gives a different interpretation of a solution to a system of linear equations. A solution \((x_1, x_2, x_3)\) specifies a \textit{linear combination} of the coefficient vectors that results in the RHS vector. Here is an example of how these vectors can combine (which, incidentally, does \textit{not} result in the
Figure 4: Coefficient vectors (red, green, blue) and RHS vector (black) in Equation 3.

RHS vector; see Figure 5a).

\[
\begin{bmatrix}
2 \\
4 \\
-2
\end{bmatrix}
\times
3
+
\begin{bmatrix}
1 \\
-6 \\
7
\end{bmatrix}
\times
2
+
\begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix}
\times
4
\]

\[
=
\begin{bmatrix}
2 \times 3 \\
4 \times 3 \\
-2 \times 3
\end{bmatrix}
+
\begin{bmatrix}
1 \times 2 \\
-6 \times 2 \\
7 \times 2
\end{bmatrix}
+
\begin{bmatrix}
1 \times 4 \\
0 \times 4 \\
2 \times 4
\end{bmatrix}
\]

\[
=
\begin{bmatrix}
12 \\
0 \\
16
\end{bmatrix}
\]

This can be verified visually.
Figure 5: (a) A linear combination of the coefficient vectors from Equation 3 (magenta) that does not result in the RHS vector (black). (b) A linear combination (dashed magenta) that does result in the RHS vector.

\begin{verbatim}
axis([0,2,-10,10,-5,10]);
v1 = [2;4;-2];
v2 = [1;-6;7];
v3 = [1;0;2];
RHS = [5;-2;9];
drawvector(RHS, 'k');
hold on;
drawvector(v1*3 + v2*2 + v3*4, 'm'); % magenta
hold off;
\end{verbatim}

The solution that gives a linear combination of the coefficient vectors resulting in the RHS vector is, as we already know, \((x_1, x_2, x_3) = (1,1,2)\). To verify this, we can plot this linear combination (Figure 5b).

\begin{verbatim}
axis([0,5,-4,0,0,10]);
drawvector(RHS, 'k');
hold on;
drawvector(v1 + v2 + v3*2, 'm--'); % dashed line
hold off;
\end{verbatim}
3 Some exercises

Exercise 1. In the same figure, plot the three planes corresponding to the equations in (4).

\begin{align*}
x_1 + 3x_2 - x_3 &= 1 \\
3x_1 + 4x_2 - 4x_3 &= 7 \\
3x_1 + 6x_2 + 2x_3 &= -3
\end{align*}

(4)

In a separate figure, plot the intersection of the first two planes, together with the intersection of the last two planes. Using these plots, determine the solution to the system of equations. Verify the solution by plotting the corresponding linear combination of the coefficient vectors in (4), together with the RHS vector (similar to Figure 5b).

Exercise 2. Make the first two plots as in Exercise 1, but now instead using the system of equations in (5).

\begin{align*}
x_1 + x_2 - 3x_3 &= -5 \\
-5x_1 - 2x_2 + 3x_3 &= 7 \\
3x_1 + x_2 - x_3 &= -3
\end{align*}

(5)

Set the axes to show $-0.5 \leq x_1 \leq 0$, $-1 \leq x_2 \leq 0.5$, and $0.5 \leq x_3 \leq 3.5$. Describe what results.

4 Singular systems of equations

Each of the systems of linear equations in (1), (2), and (4) had a unique solution. That is, in the row view, the three corresponding planes intersect at a single point; in the column view, the linear combination of the coefficient vectors that results in the RHS vector is unique. Unfortunately, this happy situation is not always the case.

Exercise 3. Consider the following system of linear equations.

\begin{align*}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + 3x_3 &= 5 \\
3x_1 + x_2 + 4x_3 &= 6
\end{align*}

(6)

Plot the three planes corresponding to the equations. Set the axes to show $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$, and $0 \leq x_3 \leq 2$. Do these planes intersect? In a separate figure, plot the intersection of the first and second planes, together with the intersection of the first and third planes. Do these lines intersect?
The systems of equations in (5) and (6) are called singular; this means they do not have a unique solution. The last two exercises show what can happen in the row view of singular systems of equations. (There are other examples of singular systems; see the textbook for these.) What happens in the column view of singular systems of equations?

Consider the coefficient vectors \((v_1, v_2, \text{ and } v_3)\) and RHS vector \((b)\) for the system of equations in (5),

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}, \\
b &= \begin{bmatrix} -5 \\ 7 \\ -3 \end{bmatrix}.
\end{align*}
\]

Recall that we can view (5) as a single vector equation

\[
v_1x_1 + v_2x_2 + v_3x_3 = b.
\]

Let’s compute the following linear combination of \(v_1, v_2, \text{ and } v_3\):

\[
\frac{-7}{18}v_1 - \frac{4}{9}v_2 + \frac{25}{18}v_3.
\]

This is easily done by hand, or in Matlab.

\[
\begin{align*}
v1 &= [1; -5; 3]; \\
v2 &= [1; -2; 1]; \\
v3 &= [-3; 3; -1];
\end{align*}
\]

\[
(-7/18)*v1-(4/9)*v2+(25/18)*v3 \ % \ no \ semicolon
\]

You should find that \((x_1, x_2, x_3) = (-7/18, -4/9, 25/18)\) is a solution to Equation (7). Now let’s compute three other linear combinations:

\[
v_1 - 6v_2, \quad \frac{1}{2}v_1 + \frac{3}{2}v_3, \quad -2v_2 + v_3.
\]

It turns out that these also give solutions to Equation (7). Therefore Equation (7) does not have a unique solution.

**Exercise 4.** Plot the coefficient vectors for the equations in (5). In another figure, plot the coefficient vectors for the equations in (6). And in yet another figure, make the same plot for (6), except change the six in the final equation to a seven. What do you notice in common about these three sets of vectors? (You may have to rotate the figures a bit to see this.) In each of these three figures, also plot the RHS vectors. What do you notice in common about the first and third of these figures, that differs from the second figure? (Hint: In the first figure, you should be able to rotate the figure so that all of the vectors seem to line up on a straight line. What does this mean? The vectors are clearly not co-linear.)