1 Definition

A vector space $V$ is a collection of vectors closed under addition and scalar multiplication; the operations of addition and scalar multiplication must satisfy the following properties:

1. Commutativity of addition:
   
   $u + v = v + u$ for all $u, v \in V$

2. Associativity of addition:
   
   $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$

3. Identity element of addition:
   
   There exists $0 \in V$ such that $u + 0 = u$ for all $u \in V$

4. Additive inverses:
   
   For each $u \in V$, there exists a unique $-u \in V$ such that $u + (-u) = 0$

5. Identity element of scalar multiplication:
   
   $1u = u$ for all $u \in V$

6. Associativity of scalar multiplication:
   
   $(c_1c_2)u = c_1(c_2u)$ for all $c_1, c_2 \in \mathbb{R}$ and $u \in V$

7. Distributivity of scalar multiplication with respect to vector addition:
   
   $c(u + v) = cu + cv$ for all $c \in \mathbb{R}$ and $u, v \in V$

8. Distributivity of scalar multiplication with respect to scalar addition:
   
   $(c_1 + c_2)u = c_1u + c_2u$ for all $c_1, c_2 \in \mathbb{R}$ and $u \in V$

A non-empty subset $W \subseteq V$ is a subspace of $V$ if $W$ is closed under addition and scalar multiplication.
2 Examples

Exercise 1. Let \( V \) be the set of polynomials in \( x \) on the interval \([-1, 1]\) of degree at most 3, i.e. \( V \) is the set of functions \( f : [-1, 1] \to \mathbb{R} \) that can be written \( f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \) for some scalars \( a_0, a_1, a_2, a_3 \in \mathbb{R} \).

Let \( f \in V \) be given by \( f(x) = 4x^3 + 3x^2 + 2x + 1 \) and \( g \in V \) be given by \( g(x) = x^3 - x^2 - 1 \).

(a) What is \( f + g \)? Is it in \( V \)?

(b) What is \( 3f \)? Is it in \( V \)?

(c) What is the additive inverse of \( g \)? Is it in \( V \)?

(d) Depict \( f, g, \) and \( 3f - g \) together on a single plot. Label each vector.

Hints: Plot the functions over a fine range of values in the interval \([-1, 1]\) (e.g. \( x = -1:0.01:1 \)). To perform element-wise exponentiation, use the \( .^\ast \) operator (e.g. \( x .^\ast 2 \); no space between the period and carat).

Check for yourself that \( V \) is indeed a vector space.

Exercise 2. Let \( W = \{f \in V : f'(0) = 0\} \) (where \( V \) is defined in Exercise 1, and \( f'(c) \) is the first derivative of \( f \) evaluated at \( c \in \mathbb{R} \)). In other words, \( W \) are the set of polynomials in \( V \) that are “flat” at 0.

(a) Let \( f \in V \) and \( g \in V \) be as defined in Exercise 1. Is \( f \in W \)? Is \( g \in W \)?

(b) Show that \( W \) is closed under addition and scalar multiplication. Conclude that \( W \) is a subspace of \( V \).

(c) Let \( h(x) = 2x^2 + 1 \). (Check for yourself that \( h \in W \).) Depict \( g, h, \) and \( 2g + h \) together on a single plot. Label each vector.

(d) True or false: Each \( f \in W \) can be written as a linear combination of \( f_0(x) = 1, f_2(x) = x^2/2, \) and \( f_3(x) = x^3/3 \).
3 Linear independence and bases

If \( c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0 \) only if \( c_1 = c_2 = \ldots = c_k = 0 \), then \( v_1, v_2, \ldots, v_k \) are linearly independent. (Otherwise, they are linearly dependent, and one of them is a linear combination of the others.) The span of \( v_1, v_2, \ldots, v_k \) is the set of all linear combinations of \( v_2, v_2, \ldots, v_k \). A set \( B \subseteq V \) is a basis for vector space \( V \) if (1) \( B \) is linearly independent, and (2) the span of \( B \) is \( V \). If \( B \) is a basis for \( V \), and the number of vectors in \( B \) is \( d \), then we say \( V \) has dimension \( d \).

How does one test if vectors \( v_1, v_2, \ldots, v_k \) are linearly independent? Stack them side-by-side in a matrix

\[
A = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}
\]

and check if the nullspace \( N(A) \) is different from \( \{0\} \). If \( N(A) \) contains some \( x = (x_1, x_2, \ldots, x_k) \neq 0 \), then this is evidence that \( v_1, v_2, \ldots, v_k \) are linearly dependent. (Do you see why?)

Exercise 3. Are “most” collections of \( n \) vectors in \( \mathbb{R}^n \) linearly independent? Test this experimentally by generating \( n \) random vectors in \( \mathbb{R}^n \) (use \texttt{randn} to generate the random vectors) and applying the above test of linear independence. Do this for \( n = 3 \), and repeat the experiment 100 times. Report how often the random set of vectors form a basis. Do things change if you move from \( n = 3 \) to \( n = 30 \)?

Exercise 4. Let \( V \) and \( W \) be the vector spaces defined in Exercises 1 and 2.

(a) Give an example of a basis \( B \) for \( W \).

(b) Name a vector \( v \in V \) so that \( B \cup \{v\} \) is a basis for \( V \).

(c) Give another basis for \( V \) (different from the one above) that includes the vector \( f(x) = x^3 + x^2 + x + 1 \).

(d) Vectors in \( V \) and \( W \) are not the usual kinds of vectors in \( \mathbb{R}^n \), so it doesn’t immediately make sense to stack them side-by-side in a matrix \( A \). How can one perform the test of linear independence on a collection of vectors in \( V \)? (Hint: If \( B \) is a basis for \( V \), then every vector in \( V \) can be written uniquely as a linear combination of basis vectors in \( B \).)

Check for yourself, using this test, that the bases you propose in the various parts of this exercise are indeed linearly independent.
(e) Let \( X = \{ f \in V : f(1) = 0 \} \). Give a basis for \( X \).

Depict the basis vectors together on a single plot. Label each vector.

(f) For \( c \in \mathbb{R} \), let \( f_c(x) = cx^3 + cx^2 + cx + c \). Let \( Y = \{ f_c \in V : c \in \mathbb{R} \} \). Give a basis for \( Y \).

Depict \( f_c \) for \(-1 \leq c \leq 1\) together on a single plot (use about 20 or so values of \( c \)). This depicts a “line segment” in the space \( V \).

For \( c \in \mathbb{R} \), let \( g_c(x) = cx^3 - cx^2 + cx - c \). Depict \( g_c \) for \(-1 \leq c \leq 1\) together on a single plot (or on the same plot with the \( f_c \)'s, but with a different color or line type). This is also a “line segment” in \( V \).

(g) What are the dimensions of \( V \), \( W \), \( X \), and \( Y \)?

(h) A function \( f \) is even if \( f(x) = f(-x) \) for all \( x \). Let \( Z = \{ f \in V : f \) is even\}. Is \( Z \) a subspace of \( V \)? If so, give a basis for it. If not, give a brief reason.

4 Misc.

Exercise 5. Which of the following sets are closed under ordinary addition?

(a) \( V = \{ \text{odd integers} \} \)

(b) \( V = \{ \text{even integers} \} \)

(c) \( V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are negative} \right\} \)

(d) \( V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is non-singular} \right\} \)

Exercise 6. There are vector spaces whose notion of addition and scalar multiplication are not the usual ones we’re used to. For the following vector space \( V \), we’ll use “\( \oplus \)” denote the special “addition operation” for \( V \) (e.g. \( v \oplus w \)) and “\( \otimes \)” to denote the special “scalar multiplication operation” for \( V \) (e.g. \( 3 \otimes v \)).
Let $V = \{ x \in \mathbb{R} : x > 0 \}$ be the set of positive real numbers. Addition and scalar multiplication in $V$ are defined as follows. For $u, v \in V$, define $u \oplus v$ to be $u \times v$, where “×” is the usual multiplication operation (e.g. $3 \oplus 4 = 3 \times 4 = 12$). For $v \in V$ and $c \in \mathbb{R}$, define $c \otimes v$ to be $v^c$, meaning “$v$ raised to the power of $c$” in the usual sense (e.g. $4 \otimes 3 = 3^4 = 81$; note that here, $3 \in V$ is the “vector” and $4 \in \mathbb{R}$ is the “scalar”).

(a) What is the “zero element” in $V$?

Hint: It is not the real number $0 \in \mathbb{R}$, because $0$ is not positive and therefore $0 \not\in V$. But there is another element in $V$, which we’ll denote by $z \in V$, that serves as the “zero element” in that $v \oplus z = z \oplus v = v$ for every $v \in V$.

(b) Show that for every $u, v \in V$, we have $u \oplus v \in V$. That is, show that $V$ is closed under “⊕ addition”.

(c) Show that for every $v \in V$ and $c \in \mathbb{R}$, we have $c \otimes v \in V$. That is, show that $V$ is closed under “⊗ scalar multiplication”.

(d) Show that, for each $v \in V$, there is an element $w \in V$ such that $v \oplus w = z$, where $z$ is the special “zero element”. That is, show that $V$ is closed under “additive inverses”.