The Context Algorithm

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Outline

Review
Outline

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Fixed Length Markov Models
Outline

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Variable Length Markov Model (VMM)
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Assigning weights to trees
Outline

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Learning the structure, an inefficient solution
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Efficient Implementation
The online Bayes Algorithm

- **Total loss** of expert \( i \)

\[
L_i^t = - \sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0
\]
The online Bayes Algorithm

- **Total loss** of expert $i$

$$L_i^t = - \sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0$$

- **Weight** of expert $i$

$$w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)$$
The online Bayes Algorithm

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  \[ w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s) \]

- **Freedom to choose initial weights.**

  \[ w_i^1 \geq 0, \quad \sum_{i=1}^{n} w_i^1 = 1 \]
The online Bayes Algorithm

- **Total loss** of expert $i$

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  L_i^t = - \sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0
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  w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)
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- Freedom to choose initial weights.

  \[
  w_i^1 \geq 0, \sum_{i=1}^{n} w_i^1 = 1
  \]

- **Prediction** of algorithm $A$

  \[
  p_A^t = \frac{\sum_{i=1}^{N} w_i^t p_i^t}{\sum_{i=1}^{N} w_i^t}
  \]
Cumulative loss vs. Final total weight

Total weight: \( W^t = \sum_{i=1}^{N} w^t_i \)
Cumulative loss vs. Final total weight

Total weight: \( W^t = \sum_{i=1}^{N} w_i^t \)

\[
\frac{W^{t+1}}{W^t} = \frac{\sum_{i=1}^{N} w_i^t e^{\log p_i^t(c_t)}}{\sum_{i=1}^{N} w_i^t}
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$$- \log \frac{W_{t+1}}{W_t} = - \log p_A^t(c^t)$$
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\[-\log \frac{W^{T+1}}{W^1} = - \sum_{t=1}^{T} \log p_A^t(c^t)\]
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**EQUALITY not bound!**
Simple Bound

- Use non-uniform initial weights $\sum_i w_i^1 = 1$
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The Context Algorithm

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= -\log \sum_{i=1}^{N} w_i^1 e^{-L_i^T}
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$$= \min_i \left(L_i^T - \log w_i^1\right)$$
A fixed length Markov Model
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- Observe a binary sequence.
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- \( X_1, \ldots, X_{t-1} \)
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- $x_1, \ldots, x_{t-1}$
- Predict next bit from past
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- $P(x_t = 1 | x_{t-1}, x_{t-2}, \ldots, x_1)$
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- Markov model of order \( k \)
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Learning a markov distribution

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  - $a_{y_1,\ldots,y_k}$ = number of times $x_{t-1} = y_1, \ldots, x_{t-k} = y_k$ and $x_t = 0$

Prediction (using Kritchevski Trofimov)

$$p(x_t = 1 | x_{t-1} = y_1, \ldots, x_{t-k} = y_k) = \frac{a_{y_1,\ldots,y_k} + 1}{2(a_{y_1,\ldots,y_k} + b_{y_1,\ldots,y_k})}$$

Total regret is at most $2k - 1 \log T$
Learning a Markov distribution

- Each tree leaf is associated with a binary sequence $y_1, \ldots, y_k$
- For each leaf keep two counters:
  - $a_{y_1, \ldots, y_k} =$ number of times $x_{t-1} = y_1, \ldots, x_{t-k} = y_k$ and $x_t = 0$
  - $b_{y_1, \ldots, y_k} =$ number of times $x_{t-1} = y_1, \ldots, x_{t-k} = y_k$ and $x_t = 1$
- Prediction (using Kritchevski Trofimov): $p(x_t = 1 | x_{t-1} = y_1, \ldots, x_{t-k} = y_k) = \frac{b_{y_1, \ldots, y_k} + 1}{2a_{y_1, \ldots, y_k} + b_{y_1, \ldots, y_k} + 1}$

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p(x_t = 1 | x_{t-1} = y_1, \ldots, x_{t-k} = y_k) = \frac{b_{y_1,\ldots,y_k} + 1/2}{a_{y_1,\ldots,y_k} + b_{y_1,\ldots,y_k} + 1}
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  - $a_{y_1,\ldots,y_k} =$ number of times $x_{t-1} = y_1, \ldots, x_{t-k} = y_k$ and $x_t = 0$
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- Total regret is at most $2^{k-1} \log T$
How variable length markov can reduce regret
How variable length markov can reduce regret

- Reducing number of leaves from 8 to 4 means reducing regret from $4 \log T$ to $2 \log T$
- English example: BAAROQUE
- When we have little data, we can get better prediction even if the children are not exactly the same.
How variable length markov can reduce regret

Reducing the number of leaves from 8 to 4 means reducing regret from $4 \log T$ to $2 \log T$. An English example:

```
B A R O Q U E
```

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English example: BAQROUE

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Reducing number of leaves from 8 to 4 means reducing regret from 4 log $T$ to 2 log $T$.

English example: B A R O Q U E

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How variable length markov can reduce regret

The Context Algorithm

Variable Length Markov Model (VMM)

Reducing number of leaves from 8 to 4 means reducing regret from $4 \log T$ to $2 \log T$.

English example: B A R Q U E

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  \[ B \]
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How variable length markov can reduce regret

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Prefix trees

- In a prefix binary tree each node has either 0 or 2 children.
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- A variable length Markov model corresponds to a prefix tree.
Prefix trees

- In a prefix binary tree each node has either 0 or 2 children.
- A node with 1 child means that some past histories are not covered.
- A variable length markov model corresponds to a prefix tree.
- But we don’t know which prefix tree to use!
Assigning probabilities to complete sequences

- Using the chain rule, we can use a prediction rule to assign probabilities to a complete sequence.

\[ P(x_1 = y_1, \ldots, x_T = y_T) = p(x_1 = y_1)p(x_2 = y_2|x_1 = y_1) \ldots \]
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- We can translate probabilities for complete sequences back into predictions.

\[ p(x_t = 1|x_1 = y_1, \ldots, x_{t-1} = y_{t-1}) = \frac{p(x_1 = y_1, \ldots, x_{t-1} = y_{t-1}, x_t = 1)}{p(x_1 = y_1, \ldots, x_{t-1} = y_{t-1})} \]
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\]

- Will come in handy soon!
Using online Bayes to learn the structure

- We assign to each tree an initial weight of $2^{-n}$ where $n$ is the number of nodes in the prefix tree.
Using online Bayes to learn the structure

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- We combine the predictions of the trees using online Bayes.
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- The total regret would be $\frac{l}{2} \log T + n$ where $l$ is the number of leaves in the prefix tree.
Using online Bayes to learn the structure

- We assign to each tree an initial weight of $2^{-n}$ where $n$ is the number of nodes in the prefix tree.
- We combine the predictions of the trees using online Bayes.
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- The papers do things slightly differently because they bound the depth of the tree by $k$. 
Using online Bayes to learn the structure

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- This algorithm maintains a weight for each tree.
Using online Bayes to learn the structure

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- The papers do things slightly differently because they bound the depth of the tree by $k$.
- This algorithm maintains a weight for each tree.
- Requires maintaining $O(2^l)$ weights!
Efficient implementation

- **First idea:** Estimate probabilities of complete sequences and use conditional to generate predictions.
Efficient implementation

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- The prior weights are used for averaging the complete sequence probabilities - they don’t need to be updated.
Efficient implementation

- **First idea:** Estimate probabilities of complete sequences and use conditional to generate predictions.
- The prior weights are used for averaging the complete sequence probabilities - they don’t need to be updated.
- **Second idea:** Compute the average over the prior efficiently.
Efficient generation of prior

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- Defines a distribution over all prefix trees.
Efficient generation of prior

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- For each node flip a fair coin.
  - Heads: Set node to be a leaf (0 children)
  - Tails: Create 2 children nodes to the node.
- Defines a distribution over all prefix trees.
- Probability of a tree with $n$ nodes is $2^{-n}$
Efficient averaging over the prior (observations)

- Maintain a KT estimator at each node of the tree.
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- Each node is visited on a subset of the iterations.
Efficient averaging over the prior (observations)

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- Allocate counters only once node is visited.
- At iteration $t$ only $t$ counters need to be updated.
- Only $k$ counters if depth of tree is bounded.
- Each node is visited on a subset of the iterations.
- Subset corresponding to node is contained in subset corresponding to node’s parent.
Efficient averaging over the prior (procedure)

- $P_e(a_s, b_s)$ the probability that the KT estimator at node $s = \langle y_1, .., y_k \rangle$ assigns to it’s subsequence of the past.
Efficient averaging over the prior (procedure)

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- $P_w^s$ the average probability assigned to the subsequence over all VMM rooted at the node $s$
Efficient averaging over the prior (procedure)

- $P_e(a_s, b_s)$ the probability that the KT estimator at node $s = \langle y_1, \ldots, y_k \rangle$ assigns to its subsequence of the past.

- $P^s_w$ the average probability assigned to the subsequence over all VMM rooted at the node $s$.

- \[
P^s_w = \frac{P_e(a_s, b_s) + P^0_w P^1_s}{2}
\]
Efficient averaging over the prior (procedure)

- $P_e(a_s, b_s)$ the probability that the KT estimator at node $s = \langle y_1, \ldots, y_k \rangle$ assigns to its subsequence of the past.

- $P_w^s$ the average probability assigned to the subsequence over all VMM rooted at the node $s$.

- $P_w^s = \frac{P_e(a_s, b_s) + P_w^0 P_w^1}{2}$

- Average probability assigned by the complete tree is $P_w^\lambda$ where $\lambda$ is the root node.