Combining infinite sets of experts

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Outline

Review
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The Universal prediction machine
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The biased coins set of experts
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Bayes using Jeffrey’s prior
   Laplace Approximation
   Choosing the optimal prior
   Kritchevski Trofimov Prediction Rule
   Laplace Rule of Succession
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Generalization to larger sets of distributions
The online Bayes Algorithm

- **Total loss** of expert $i$

\[
L^t_i = - \sum_{s=1}^{t} \log p^s_i(c^s); \quad L^0_i = 0
\]
The online Bayes Algorithm

- **Total loss** of expert $i$

$$L_i^t = - \sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0$$

- **Weight** of expert $i$

$$w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)$$
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$$w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)$$

- **Freedom to choose initial weights.**

$$w_i^1 \geq 0, \quad \sum_{i=1}^{n} w_i^1 = 1$$
The online Bayes Algorithm

- **Total loss** of expert $i$
  \[ L_i^t = - \sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0 \]

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- Freedom to choose initial weights.
  \[ w_i^1 \geq 0, \quad \sum_{i=1}^{n} w_i^1 = 1 \]

- **Prediction** of algorithm $A$
  \[ p_A^t = \frac{\sum_{i=1}^{N} w_i^t p_i^t}{\sum_{i=1}^{N} w_i^t} \]
Cumulative loss vs. Final total weight

Total weight: $W^t = \sum_{i=1}^{N} w_i^t$
Cumulative loss vs. Final total weight

Total weight: \( W^t \dot{=} \sum_{i=1}^{N} w^t_i \)

\[
\frac{W^{t+1}}{W^t} = \frac{\sum_{i=1}^{N} w^t_i e^{\log p_i^t(c^t)}}{\sum_{i=1}^{N} w^t_i}
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\[- \log \frac{W^{t+1}}{W^t} = - \log p_A^t(c^t)\]
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Equality not bound!
Simple Bound

- Use non-uniform initial weights $\sum_i w_i^1 = 1$
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\[ = -\log \sum_{i=1}^N w_i^1 e^{-L_i^T} \]
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$$= \min_i \left( L_i^T - \log w_i^1 \right)$$
Standardizing online prediction algorithms

- Fix a universal Turing machine $U$.
Infinite sets of experts

The Universal prediction machine

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- An online prediction algorithm $E$ is a program that
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- Any online prediction algorithm can be represented as code $\vec{b}(E)$ for $U$. The code length is $|\vec{b}(E)|$. 
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- Most sequences do not correspond to valid prediction algorithms.
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- Most sequences do not correspond to valid prediction algorithms.
- $V(\vec{b}, \vec{X}, t) = 1$ if the program $\vec{b}$, given $\vec{X}$ as input, halts within $t$ steps and outputs a well-formed prediction. Otherwise $V(\vec{b}, \vec{X}, t) = 0$
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- $V(\vec{b}, \vec{X}, t) = 1$ if the program $\vec{b}$, given $\vec{X}$ as input, halts within $t$ steps and outputs a well-formed prediction. Otherwise $V(\vec{b}, \vec{X}, t) = 0$
- $V(\vec{b}, \vec{X}, t)$ is computable (recursively enumerable).
Assign to the code $\vec{b}$ the initial weight $w^1_{\vec{b}} = 2^{-|\vec{b}| - \log_2 |\vec{b}|}$. 

The total initial weight over all finite binary sequences is one. Run the Bayes algorithm over "all" prediction algorithms. Technical details: On iteration $t$, $|\vec{X}| = t$. Use the predictions of programs $\vec{b}$ such that $|\vec{b}| \leq t$ and for which $V(\vec{b}, \vec{X}, 2^t) = 1$. Assign the remaining mass the prediction $1/2$ (insuring a loss of $1$).
A universal prediction machine

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Infinite sets of experts

The Universal prediction machine

Performance of the universal prediction algorithm

- Using \( L_A \leq \min_i (L_i - \log w_i) \)
Performance of the universal prediction algorithm

- Using $L_A \leq \min_i (L_i - \log w_i)$
- Assume $E$ is a prediction algorithm which generates the $t$th prediction in time smaller than $2^t$
Infinite sets of experts

The Universal prediction machine

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- Using $L_A \leq \min_i (L_i - \log w_i)$
- Assume $E$ is a prediction algorithm which generates the $t$th prediction in time smaller than $2^t$
- When $t \leq |\bar{b}(E)|$ the algorithm is not used and thus its loss is 1

Code length is arbitrarily close to the Kolmogorov complexity of the sequence.

Ridiculously bad running time.
Infinite sets of experts
↓
The Universal prediction machine

Performance of the universal prediction algorithm

- Using $L_A \leq \min_i (L_i - \log w_i^1)$
- Assume $E$ is a prediction algorithm which generates the $t$th prediction in time smaller than $2^t$
- When $t \leq |\bar{b}(E)|$ the algorithm is not used and thus it’s loss is 1
- We get that the loss of the Universal algorithm is at most $2|\bar{b}(E)| + \log_2 |\bar{b}(E)| + L_E$
Infinite sets of experts

Performance of the universal prediction algorithm

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- Assume \( E \) is a prediction algorithm which generates the \( t \)th prediction in time smaller than \( 2^t \)
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- More careful analysis can reduce \( 2|\vec{b}(E)| + \log_2 |\vec{b}(E)| \) to \( |\vec{b}(E)| \)
Performance of the universal prediction algorithm

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Infinite sets of experts

The Universal prediction machine

Performance of the universal prediction algorithm

- Using $L_A \leq \min_i (L_i - \log w_i^1)$
- Assume $E$ is a prediction algorithm which generates the $t$th prediction in time smaller than $2^t$
- When $t \leq |\tilde{b}(E)|$ the algorithm is not used and thus it’s loss is 1
- We get that the loss of the Universal algorithm is at most $2|\tilde{b}(E)| + \log_2 |\tilde{b}(E)| + L_E$
- More careful analysis can reduce $2|\tilde{b}(E)| + \log_2 |\tilde{b}(E)|$ to $|\tilde{b}(E)|$
- Code length is arbitrarily close to the Kolmogorov Complexity of the sequence.
- Ridiculously bad running time.
Infinite sets of experts

The Universal prediction machine

Bayes coding is better than two part codes

- Simple bound as good as bound for two part codes (MDL) but enables online compression

Suppose we have $K$ copies of each expert.

Two part code has to point to one of the $K_N$ experts

$$L_A = \log K + \min_i L_T i = \log N + \min_i L_T i$$

If we use Bayes predictor + arithmetic coding we get:

$$L_A = -\log W_T + 1 \leq \log K + \max_i 1_N k e^{-L_T i} = \log N + \min_i L_T i$$

We don't pay a penalty for copies.

More generally, the regret is smaller if many of the experts perform well.
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- Suppose we have $K$ copies of each expert.
- Two part code has to point to one of the $KN$ experts

$$L_A \leq \log NK + \min_i L^T_i = \log NK + \min_i L^T_i$$

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  $$L_A \leq \log NK + \min_i L_i^T = \log NK + \min_i L_i^T$$
- If we use Bayes predictor + arithmetic coding we get:
  $$L_A = -\log W^{T+1} \leq \log K \max_i \frac{1}{NK} e^{-L_i^T} = \log N + \min_i L_i^T$$

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- Suppose we have \( K \) copies of each expert.
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- If we use Bayes predictor + arithmetic coding we get:
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- We don’t pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.
The biased coins set of experts

- Each expert corresponds to a biased coin, predicts with a fixed $\theta \in [0, 1]$. 
The biased coins set of experts

- Each expert corresponds to a biased coin, predicts with a fixed $\theta \in [0, 1]$.
- Set of experts is uncountably infinite.
The biased coins set of experts

- Each expert corresponds to a biased coin, predicts with a fixed $\theta \in [0, 1]$.
- Set of experts is **uncountably infinite**.
- Only countably many experts can be assigned non-zero weight.
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- Each expert corresponds to a biased coin, predicts with a fixed $\theta \in [0, 1]$.
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- Instead, we assign the experts a Density Measure.
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- Only countably many experts can be assigned non-zero weight.
- Instead, we assign the experts a Density Measure.
- $L_A \leq \min_i (L_i - \log w_i^1)$ is meaningless.
Infinite sets of experts

The biased coins set of experts

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- Each expert corresponds to a biased coin, predicts with a fixed \( \theta \in [0, 1] \).
- Set of experts is uncountably infinite.
- Only countably many experts can be assigned non-zero weight.
- Instead, we assign the experts a Density Measure.
- \( L_A \leq \min_i (L_i - \log w_i^1) \) is meaningless.
- Can we still get a meaningful bound?
Bayes Algorithm for biased coins

- Replace the initial weight by a density measure
  \[ w(\theta) = w^1(\theta), \int_0^1 w(\theta) d\theta = 1 \]
Bayes Algorithm for biased coins

- Replace the initial weight by a density measure
  \[ w(\theta) = w^1(\theta), \quad \int_0^1 w(\theta) d\theta = 1 \]

- Relationship between final total weight and total log loss remains unchanged:

  \[ L_A = \ln \int_0^1 w(\theta) e^{-\sum_{T+1}^{T+1} \theta} d\theta \]
Bayes Algorithm for biased coins

- Replace the initial weight by a density measure $w(\theta) = w^1(\theta)$, $\int_0^1 w(\theta) d\theta = 1$

- Relationship between final total weight and total log loss remains unchanged:

$$L_A = \ln \int_0^1 w(\theta) e^{-L^T_\theta + 1} d\theta$$

- We need a new lower bound on the final total weight
Main Idea

If $w^t(\theta)$ is large then $w^t(\theta + \epsilon)$ is also large.
Main Idea

If $w^t(\theta)$ is large then $w^t(\theta + \epsilon)$ is also large.
Expanding the exponent around the peak

- For log loss the best $\hat{\theta}$ is empirical distribution of the seq.

\[
\hat{\theta} = \sum_{1 \leq t \leq T} \frac{x^t = 1}{T}
\]
Expanding the exponent around the peak

For log loss the best \( \theta \) is empirical distribution of the seq.

\[
\hat{\theta} = \frac{\# \{ x^t = 1; 1 \leq t \leq T \}}{T}
\]

The total loss scales with \( T \)

\[
L_\theta = T \cdot (\hat{\theta} \ell(\theta, 1) + (1 - \hat{\theta}) \ell(\theta, 0)) \equiv T \cdot g(\hat{\theta}, \theta)
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Expanding the exponent around the peak

- For log loss, the best $\theta$ is the empirical distribution of the sequence.
  \[
  \hat{\theta} = \frac{\#\{x^t = 1; 1 \leq t \leq T\}}{T}
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- The total loss scales with $T$
  \[
  L_\theta = T \cdot (\hat{\theta} \ell(\theta, 1) + (1 - \hat{\theta}) \ell(\theta, 0)) = T \cdot g(\hat{\theta}, \theta)
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$$\hat{\theta} = \frac{\#\{x^t = 1; 1 \leq t \leq T\}}{T}$$

- The total loss scales with $T$

$$L_\theta = T \cdot (\hat{\theta} \ell(\theta, 1) + (1 - \hat{\theta}) \ell(\theta, 0)) \doteq T \cdot g(\hat{\theta}, \theta)$$

$$L_A - L_{\text{min}} \leq \ln \int_0^1 w(\theta) e^{-L_\theta} d\theta - \ln e^{L_{\text{min}}}$$
Infinite sets of experts

Bayes using Jeffrey’s prior

Laplace Approximation

Expanding the exponent around the peak

- For log loss the best $\theta$ is empirical distribution of the seq.

$$\hat{\theta} = \frac{\# \{ x^t = 1; \ 1 \leq t \leq T \}}{T}$$

- The total loss scales with $T$

$$L_\theta = T \cdot (\hat{\theta} \ell(\theta, 1) + (1 - \hat{\theta}) \ell(\theta, 0)) \approx T \cdot g(\hat{\theta}, \theta)$$

$$L_A - L_{\min} \leq \ln \int_0^1 w(\theta) e^{-L_\theta} d\theta - \ln e^{L_{\min}}$$

$$= \ln \int_0^1 w(\theta) e^{-(L_\theta - L_{\min})} d\theta$$

$$pause = \ln \int_0^1 w(\theta) e^{T(g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))} d\theta$$
Laplace approximation (idea)

- Taylor expansion of $g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta})$ around $\theta = \hat{\theta}$. 
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Laplace Approximation (details)

\[ \int_0^1 w(\theta) e^{T(g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))} d\theta \]
Laplace Approximation (details)

\[ \int_{0}^{1} w(\theta) e^{T(g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))} d\theta \]

\[ = w(\hat{\theta}) \sqrt{\frac{-2\pi}{T \left( \frac{d^2}{d\theta^2} \right)_{\theta=\hat{\theta}} (g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))}} + O(T^{-3/2}) \]
Choosing the optimal prior

Choose $w(\theta)$ to maximize the worst-case final total weight

$$
\min_{\hat{\theta}} w(\hat{\theta}) \sqrt{T \frac{d^2}{d\theta^2} \bigg|_{\theta=\hat{\theta}} (g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))} \sqrt{-2\pi}
$$
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\]

Make bound equal for all \( \hat{\theta} \in [0, 1] \) by choosing

\[
w^*(\hat{\theta}) = \frac{1}{Z} \sqrt{ \frac{\frac{d^2}{d\theta^2}\bigg|_{\theta=\hat{\theta}} (g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))}{-2\pi} },
\]

where \( Z \) is the normalization factor:

\[
Z = \sqrt{\frac{1}{2\pi}} \int_0^1 \sqrt{ \frac{d^2}{d\theta^2}\bigg|_{\theta=\hat{\theta}} (g(\hat{\theta}, \hat{\theta}) - g(\hat{\theta}, \hat{\theta})) } \ d\hat{\theta}
\]
The bound for the optimal prior

Plugging in we get

\[
L_A - L_{\text{min}} \leq \ln \int_0^1 w^*(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta
\]

\[
= \ln \left( \sqrt{\frac{2\pi Z}{T}} + O(T^{-3/2}) \right)
\]

\[
= \frac{1}{2} \ln \frac{T}{2\pi} - \frac{1}{2} \ln Z + O(1/T)
\]
Solving for log-loss

- The exponent in the integral is

\[ g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}) = \hat{\theta} \ln \frac{\hat{\theta}}{\theta} + (1 - \hat{\theta}) \ln \frac{1 - \hat{\theta}}{1 - \theta} = D_{KL}(\hat{\theta} || \theta) \]
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- The second derivative

\[ \left. \frac{d^2}{d\theta^2} D_{KL}(\hat{\theta} \parallel \theta) \right|_{\theta = \hat{\theta}} = \hat{\theta} (1 - \hat{\theta}) \]

Is called the empirical Fisher information
Solving for log-loss

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is called the empirical Fisher information.

- The optimal prior:

\[ w^*(\hat{\theta}) = \frac{1}{\pi \sqrt{\hat{\theta}(1 - \hat{\theta})}} \]

Known in general as Jeffrey’s prior. And, in this case, the Dirichlet-(1/2, 1/2) prior.
The cumulative log loss of Bayes using Jeffrey’s prior

\[ L_A - L_{\text{min}} \leq \frac{1}{2} \ln(T + 1) + \frac{1}{2} \ln \frac{\pi}{2} + O(1/T) \]
But what is the prediction rule?

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- As luck would have it the Dirichlet prior is the **conjugate prior** for the Binomial distribution.
- Observed $t$ bits, $n$ of which were 1. The posterior is:

$$
\frac{1}{Z \sqrt{\theta(1-\theta)}} \theta^n (1-\theta)^{t-n} = \frac{1}{Z} \theta^{n-1/2} (1-\theta)^{t-n-1/2}
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- The posterior average is:

$$
\frac{\int_0^1 \theta^{n+1/2} (1 - \theta)^{t-n-1/2} d\theta}{\int_0^1 \theta^{n-1/2} (1 - \theta)^{t-n-1/2} d\theta} = \frac{n + 1/2}{t + 1}
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- This is called the Trichevsky Trofimov prediction rule.
Laplace Rule of Succession

- Laplace suggested using the uniform prior, which is also a conjugate prior.
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L_A - L_{\min} = \ln T + O(1)
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- Suffers larger regret when \(\hat{\theta}\) is far from \(1/2\)
What is the optimal prediction when $T$ is known in advance?
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$$L_*^T - \min_{\theta} L_{\theta}^T \geq \frac{1}{2} \ln(T + 1) + \frac{1}{2} \ln \frac{\pi}{2} - O\left(\frac{1}{\sqrt{T}}\right)$$
Multinomial Distributions

- For a distribution over \( k \) elements (Multinomial) [Xie and Barron]
Multinomial Distributions

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- The constant \( C \) is optimal.
Exponential Distributions

- For any set of distributions from the exponential family defined by $k$ parameters (Some technical conditions on closure of set??) [Rissanen??]
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- Use Bayes Algorithm with Jeffrey’s prior:

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w^*(\hat{\theta}) = \frac{1}{Z} \frac{1}{\sqrt{H(D_{KL}(\hat{\theta}||\theta))|_{\theta=\hat{\theta}}}}
\]

\( H \) denotes the Hessian.
Exponential Distributions

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$$w^*(\hat{\theta}) = \frac{1}{Z} \frac{1}{\sqrt{\det\left(\mathbf{H}(D_{KL}(\hat{\theta} || \theta))\right)|_{\theta=\hat{\theta}}}}$$

$\mathbf{H}$ denotes the Hessian.

- $$L_A - L_{\text{min}} \leq \frac{k-1}{2} \ln T - \ln Z + o(1)$$
General Distributions

- Characterize distribution family by metric entropy.

\[ N\left(\frac{1}{\epsilon}\right) = O\left(\frac{1}{\epsilon^d}\right) \]

where \(d\) is the number of parameters. According to Haussler and Opper, the coefficient in front of \(\ln T\) is optimal for distribution families where the metric entropy is up to \(N\left(\frac{1}{\epsilon}\right) = O\left(e^{\frac{1}{2\epsilon} - \alpha}\right)\) for all \(\alpha \leq \frac{5}{2}\).
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  \]
  
  For all $\alpha \leq 5/2$. 
Infinite sets of experts

Generalization to larger sets of distributions

next Class

- Variable-length markov models - a set of distributions with increasing number of parameters.
next Class

- Variable-length markov models - a set of distributions with increasing number of parameters.
- THe context algorithm: An efficient implementation of the Bayes algorithm which achieves close-to-optimal worst case bounds.